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Memory functions and correlations in additive binary Markov chains

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Abstract
A theory of additive Markov chains with a long-range memory, proposed earlier in Usatenko *et al* (2003 *Phys. Rev.* E **68** 061107), is developed and used to describe statistical properties of long-range correlated systems. The convenient characteristics of such systems, memory functions and their relation to the correlation properties of the systems are examined. Various methods for finding the memory function via the correlation function are proposed. The inverse problem (calculation of the correlation function by means of the prescribed memory function) is also solved. This is demonstrated for the analytically solvable model of the system with a step-wise memory function.

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1. Introduction

The problem of long-range correlated dynamic systems (LRCS) has been under study for a long time in many areas of contemporary physics [1–6], biology [7–12], economics [8, 13, 14], linguistics [15–19], etc [8, 20]. One of the ways to get a correct insight into the nature of correlations in a system consists in constructing a mathematical object (for example, a correlated sequence of symbols) so that some of its statistical properties coincide with those of the initial system. This problem is closely related to the realization problem (see, for instance, [21–23]). There exist many algorithms for generating long-range correlated sequences: the inverse Fourier transformation [20, 24], the expansion-modification Li method [25], the Voss procedure of consequent random additions [26], the correlated Levy walks [27], etc [20]. The use of *multi-step Markov* chains is an essential point among them because they offer a possibility of constructing a random sequence having necessary correlated properties in the most natural way. This was demonstrated in [28], where the concept of a Markov chain with the *step-wise memory function* was introduced. The correlation properties of some dynamical systems (coarse-grained sequences of *Eukarya's DNA and dictionaries*) can be well described by this model [28].

A sequence of symbols in the Markov chain can be thought of as the sequence of hops of a certain particle so that the binary sequence corresponds to the hops of length 1 in two opposite directions at every fixed time interval. The Brownian motion of this particle is correlated. The sum of L sequential symbols in the Markov chain is the displacement of the particle during the 'time' interval L. This point makes it possible to employ statistical methods for examining the correlation properties of complex dynamic systems using mapping onto the binary Markov chains. Another important reason for exploring Markov chains is their application to various physical objects [29–31], e.g., to the Ising chains of spins. The problem of a thermodynamics description of the Ising chains with long-range spin interaction is still unresolved even for the 1D case. The association of such systems with the Markov chains can shed light on the non-extensive thermodynamics of the LRCS.

In this paper, we ascertain the relation between the memory function of the additive Markov chains and the correlation properties of the systems under consideration. We examine the simplest variant of the random sequences, dichotomic (binary) ones, although the proposed theory can be applied to arbitrary additive Markov processes with a finite or infinite number of states.

The paper is organized as follows. In the first section, we introduce general relations for the Markov chains, derive an equation connecting the correlation and memory functions of additive Markov chains and verify the robustness of our method by numerical simulations. The second part is devoted to the study of the correlation function for the Markov chain with the step-wise memory function. In subsection 3.2, we reveal a band structure of the correlation function and obtain its explicit expression. Subsection 3.3 contains the results of an asymptotic study of the correlation function.

2. General properties of additive Markov chains

2.1. Basic notions

Let us consider a homogeneous binary sequence of symbols, $a_i = \{0, 1\}, i \in \mathbf{Z} = \ldots, -2, -1, 0, 1, 2, \ldots$ To determine the *N-step Markov chain*, we have to introduce the *conditional probability* $P(a_i|a_{i-N}, a_{i-N+1}, \ldots, a_{i-1})$ of the definite symbol a_i (for example, $a_i = 1$ or $a_i = 0$) occurring after the N-word $T_{N,i}$, where $T_{N,i}$ denotes the sequence of symbols $a_{i-N}, a_{i-N+1}, \ldots, a_{i-1}$. Thus, it is necessary to define 2^N values of the P-function corresponding to each possible configuration of the symbols in the N-word $T_{N,i}$. Since we intend to deal with sequences possessing the memory length of the order of 10^6, we need to make some simplifications. Suppose that the P-function has the *additive* form

$$P(a_i = 1|T_{N,i}) = \sum_{k=1}^{N} f(a_{i-k}, k). \tag{1}$$

Here, the value $f(a_{i-k}, k)$ is the additive contribution of the symbol a_{i-k} to the conditional probability of the symbol unity occurring at the ith site. Equation (1) corresponds to the additive influence of the previous symbols on the generated one. Such a Markov chain is referred to as an *additive Markov chain* [32]. The homogeneity of the Markov chain is provided by the independence of the conditional probability equation (1) of index i. It is possible to consider equation (1) as the first term in the expansion of conditional probability in the formal series of terms that correspond to the additive (or unary), binary, ternary and such functions up to an N-ary one.

Let us rewrite equation (1) in an equivalent form

$$P(a_i = 1|T_{N,i}) = \bar{a} + \sum_{r=1}^{N} F(r)(a_{i-r} - \bar{a}). \tag{2}$$

Here,

$$\bar{a} = \sum_{r=1}^{N} f(0,r) \bigg/ \left[1 - \sum_{r=1}^{N}(f(1,r) - f(0,r))\right]$$

is the average number of unities in the sequence, [32], and

$$F(r) = f(1,r) - f(0,r).$$

We refer to $F(r)$ as the *memory function* (MF). It describes the strength of impact of the previous symbol a_{i-r} upon a generated one, a_i. Evidently, this function has to satisfy condition $0 \leqslant P(a_i = 1|T_{N,i}) \leqslant 1$. To the best of our knowledge, the concept of the memory function for multi-step Markov chains was introduced in [19, 28]. The authors indicate that it is convenient to use it in describing the correlated properties of complex dynamical systems with long-range correlations.

The function $P(a_i = 1|T_{N,i})$ contains complete information about the correlation properties of the Markov chain. In general, the correlation function and other moments are employed as input characteristics to describe the correlated random systems. Yet the correlation function takes an account of both the direct interconnection of the elements a_i and a_{i+r} and their indirect interaction via other elements. Our approach operates with the 'origin' characteristics of the system, specifically with the memory function.

The positive values of the MF result in persistent diffusion where previous displacements of the Brownian particle in some direction provoke its consequent displacement in the same direction. The negative values of the MF correspond to the antipersistent diffusion where the changes in the direction of motion are more probable. In terms of the Ising long-range particles interaction model, which could be naturally associated with the Markov chains, the positive values of the MF correspond to the attraction of particles whereas the negative ones conform to the repulsion.

We consider the distribution $W_L(k)$ of the words of a definite length L by the number k of unities in them, $k_i(L) = \sum_{l=1}^{L} a_{i+l}$, and the variance $D(L)$ of $k_i(L)$,

$$D(L) = \overline{(k - \bar{k})^2}, \tag{3}$$

where the definition of the average value of $g(k)$ is $\overline{g(k)} = \sum_{k=0}^{L} g(k) W_L(k)$.

Another statistical characteristic of random sequences is the correlation function

$$K(r) = \overline{a_i a_{i+r}} - \bar{a}^2. \tag{4}$$

By definition, the correlation function is even, $K(-r) = K(r)$, and $K(0) = \bar{a}(1 - \bar{a})$ is the variance of random variable a_i. The correlation function is related to the above-mentioned variance by the equation

$$K(r) = \tfrac{1}{2}(D(r-1) - 2D(r) + D(r+1)), \tag{5}$$

or

$$K(r) = \frac{1}{2}\frac{\mathrm{d}^2 D(r)}{\mathrm{d}r^2} \tag{6}$$

in the continuous limit.

2.2. Derivation of the main equation

In this subsection, we obtain a very important relation connecting the memory and correlation functions of the additive Markov chain. Note that this relation has been already derived by the variation method in [32]. Here, we offer a more rigorous proof of this equation that also allows one to have a better understanding of the nature of this relation.

Let us introduce the function $\phi(r) = P(a_i = 1|a_{i-r} = 1)$, which is the probability of symbol $a_i = 1$ occurring under condition that the previous symbol a_{i-r} is likewise equal to unity. This function is obviously connected to the correlation function $K(r)$, see equation (4), since the quantity $\overline{a_i a_{i-r}}$ is the probability of simultaneous equality to unity of both symbols, a_i and a_{i-r}. It can be expressed in terms of the conditional probability $\phi(r)$,

$$\overline{a_i a_{i-r}} = P(a_i = 1|a_{i-r} = 1)P(a_{i-r} = 1) = \bar{a}\phi(r). \tag{7}$$

Substituting equation (7) into equation (4), we get

$$K(r) = \bar{a}\phi(r) - \bar{a}^2. \tag{8}$$

For the N-step Markov chain, the probability of the symbol $a_i = 1$ occurring depends on the previous N-word only. Therefore, to obtain the value of $\phi(r)$ one needs to average the conditional probability P equation (2) over all realizations of the N-words at $a_{i-r} = 1$,

$$\begin{aligned}\phi(r) &= P(a_i = 1|a_{i-r} = 1)\\ &= \sum_{T_{N,i}} P(a_i = 1|T_{N,i}, a_{i-r} = 1)P(T_{N,i}|a_{i-r} = 1).\end{aligned} \tag{9}$$

If the value of r is less than or equal to N, then a_{i-r} in equation (9) is one of the symbols $a_{i-1}, a_{i-2}, \ldots, a_{i-N}$ in the word $T_{N,i}$. In this case, the summands in equation (9), with the word $T_{N,i}$ that contains the symbol zero at the $(i-r)$ th position, are equal to zero. If $r > N$, the memory function $F(r)$ equals zero in this region and, hence, the sum in equation (9) contains *all* terms corresponding to all different N-words.

Substituting equation (2) into equation (9), we have

$$\phi(r) = \bar{a}\sum_{T_{N,i}} P(T_{N,i}|a_{i-r} = 1) + \sum_{r'=1}^{N} F(r')\sum_{T_{N,i}}(a_{i-r'} - \bar{a})P(T_{N,i}|a_{i-r} = 1). \tag{10}$$

According to the normalization condition, the first sum in equation (10) is equal to unity. Consider the sum

$$\sum_{T_{N,i}} a_{i-r'}P(T_{N,i}|a_{i-r} = 1) \tag{11}$$

in the second term on the rhs of equation (10). The symbol $a_{i-r'}$ is contained within the word $T_{N,i}$. Therefore, equation (11) represents the average value of $a_{i-r'}$ when $a_{i-r} = 1$. In other words, it equals the probability $\phi(r-r')$ of $a_{i-r'} = 1$ occurring provided $a_{i-r} = 1$:

$$\sum_{T_{N,i}} a_{i-r'}P(T_{N,i}|a_{i-r} = 1) = \phi(r-r'). \tag{12}$$

Substituting this equation into equation (10), we obtain

$$\phi(r) = \bar{a} + \sum_{r'=1}^{N} F(r')(\phi(r-r') - \bar{a}). \tag{13}$$

Taking into account equation (8), we arrive at the relation between the memory function and the correlation function:

$$K(r) = \sum_{r'=1}^{N} F(r')K(r-r'), \qquad r \geqslant 1. \tag{14}$$

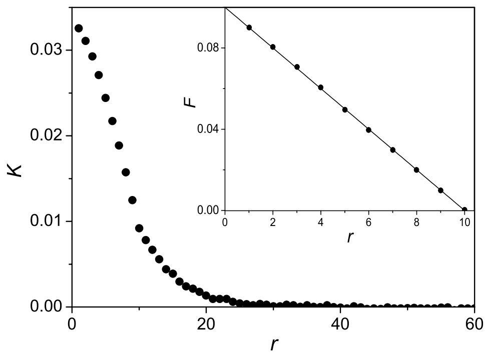

Figure 1. Calculated correlation function $K(r)$ of the Markov chain constructed with the model memory function $F(r)$, equation (16), shown by the solid line in the inset. The dots in the inset correspond to the memory function reconstructed by solving equation (14) with the correlation function $K(r)$ presented in the main panel.

This result can be understood intuitively. The memory function values $F(r)$ characterize a direct influence of symbols on the distance r. In contrast, the correlation function values $K(r)$ attest to the implicit influence as well. The convolution is just a reflection of this: every summand in equation (14) is the 'influence' of symbols on the shorter distance $(r - r')$ with 'intensity' $F(r')$.

Another equation resulting from equation (14) by double summation over index r establishes a relationship between the memory function $F(r)$ and variance $D(L)$,

$$\begin{aligned} M(r, 0) &= \sum_{r'=1}^{N} F(r')M(r, r'), \\ M(r, r') &= D(r - r') - (D(-r') + r[D(-r' + 1) - D(-r')]). \end{aligned} \tag{15}$$

Equation (5) and parity of the function $D(r)$ are used in equation (15).

The latter equation shows that it is convenient to use the variance $D(L)$ instead of the correlation function $K(r)$. The function $K(r)$, being a second derivative of $D(r)$ in continuous approximation, is less robust in computer simulations. It is a strong reason for why we prefer to use equation (15) for long-range memory sequences. This is our tool for finding the memory function $F(r)$ of a sequence using the variance $D(L)$.

2.3. Numerical reconstruction of the memory function

Let us verify the robustness of our method by numerical simulations. We consider a model *memory function*

$$F(r) = 0.1 \begin{cases} 1 - r/10 & 1 \leqslant r < 10, \\ 0 & r \geqslant 10, \end{cases} \tag{16}$$

shown in the inset in figure 1 by a solid line. Using equation (2), we construct a random unbiased $\bar{a} = 1/2$, Markov chain. Then we numerically calculate the correlation function $K(r)$ by solving the set of N linear equations (14) with the aim of the constructed binary sequence of the length 3×10^6. The result of these calculations is given in figure 1. One can see that the correlation function $K(r)$ roughly mimics the memory function $F(r)$ over the region $1 \leqslant r \leqslant 10$. In the region $r > 10$, the memory function is equal to zero but the correlation function does not vanish[3]. Then, using the obtained correlation function $K(r)$, we solve equation (14) numerically. The result is shown in the inset in figure 1 by dots. We have an excellent agreement between initial, equation (16), and reconstructed memory functions $F(r)$.

The binary correlation function does not permit one to generate the random sequence of symbols. The obtained relation (14) between the correlation and memory functions allows us to find the memory function as the solution of equation (14) and, thus, to construct a binary sequence with a *prescribed correlation function*. This is the nontrivial result obtained in this paper.

Yet another approach to numerical reconstruction by finding the memory function is an iteration procedure. For its realization, let us rewrite equation (14) in the form

$$F(r) = \frac{K(r)}{K(0)} - \sum_{r'=1, r'\neq r}^{N} \frac{K(r-r')}{K(0)} F(r'). \tag{17}$$

Using equation (17) with starting iteration $F_0(r) = 0$, we obtain

$$F_{n+1}(r) = \frac{K(r)}{K(0)} - \sum_{r'=1, r'\neq r}^{N} \frac{K(r-r')}{K(0)} F_n(r'), \qquad n \geqslant 0. \tag{18}$$

Thus, the memory function can be presented as the series

$$F(r) = \frac{K(r)}{K(0)} - \sum_{r'\neq r} \frac{K(r-r')K(r')}{K^2(0)} + \sum_{r'\neq r} \sum_{r''\neq r'} \frac{K(r-r')K(r'-r'')K(r'')}{K^3(0)} + \cdots \tag{19}$$

Note that the Markov chain with the definite correlation function $K(r)$ exists if the series (19) is convergent and the obtained function implies the probability (2) satisfying the requirement $0 \leqslant P(a_i = 1|T_{N,i}) \leqslant 1$ for the arbitrary word $T_{N,i}$. If $\bar{a} = 1/2$, we obtain the restriction $\sum |F(r)| \leqslant 1$. A sufficient, but not necessary, requirement is $\sum_{r=1}^{N-1} |K(r)| \leqslant 1/12 - |K(N)|/3$.

3. Correlation function of the chain with the step-wise memory function

In the previous section, we have derived the relationship (14) between two characteristics of the Markov chain, the memory and correlation functions, and used this equation to solve the problem of finding the memory function via the known correlation function. Here, we present the procedure of solving the inverse problem. We assume the memory function to be known and find the correlation function of the correspondent additive Markov chain. Formally, equation (14) makes it possible to obtain $K(r)$ for arbitrary $F(r)$. However, as will be seen in the following sections, a sufficiently wide range of analytic results can be found precisely for the chain with the step-wise memory function

$$F(r) = \begin{cases} \alpha & r \leqslant N, \\ 0 & r > N. \end{cases} \tag{20}$$

The restriction imposed on parameter α can be deduced from equation (2): $|\alpha| < 1/N$. Note that each of the unities in the preceding N-word promotes the emergence of new unity if $0 < \alpha < 1/N$. This corresponds to the persistent diffusion. The region of parameter α determined by inequality $-1/N < \alpha < 0$ corresponds to the antipersistent diffusion. If $\alpha = 0$, one has the case of the non-correlated Brownian motion.

[3] The existence of the 'additional tail' in the correlation function is in agreement with [19] and corresponds to the well-known fact that the correlation length is always larger than the region of memory function action.

3.1. Main equation for the correlation function

Substituting equation (20) into equation (14), we arrive at the relation

$$K(r) = \alpha \sum_{r'=1}^{N} K(r - r'), \qquad r \geqslant 1. \tag{21}$$

Here, the correlation function is assumed to be even, $K(-r) = K(r)$. Equation (21) is the linear recurrence of the order of N for $r \geqslant N + 1$, so we stand in need of N initial conditions. For the unbiased sequence, $\bar{a} = 1/2$, we have $K(0) = 1/4$. The solution of equation (21) written for $r = 1, \dots, N$ yields the constant value of the correlation function, $K(r) = K_0$, at $r = 1, \dots, N - 1$:

$$K_0 = \frac{\alpha}{4(1 - \alpha(N - 1))}. \tag{22}$$

Subtracting equation (21) from the same equation written for $r + 1$, we derive another, more convenient, form of the recurrence:

$$K(r + 1) - (1 + \alpha)K(r) + \alpha K(r - N) = 0. \tag{23}$$

This equation is of the order of $N + 1$, thus if is necessary to have an additional initial condition. It can be derived from equation (21): $K(N) = K_0$. Note that the possibility of rewriting equation (21) in the form of equation (23) is the result of the simple structure of the memory function. We solve the obtained recurrent equations by the most natural method, by means of step-by-step finding the sequent values of the correlation function. Such an approach is very suitable for the analysis of the correlation function at $r \gtrsim N$.

3.2. Correlation function at $r \gtrsim N$

3.2.1. Band structure of the correlation function. Equation (21) allows one to find numerically the unknown correlation function $K(r)$. The result of this step-by-step calculation is presented in figure 2 by a solid line. One can easily see the discontinuity of $K(r)$ at the point $L = N = 100$; the breakpoint of the curve is observed at $L = 2N$. Such a behaviour of the correlation function results from using the step-wise memory function. To clarify this fact, it is convenient to change the variable r by the band number s and the intra-band number ρ:

$$K(r) = K_s(\rho), \qquad r = sN + \rho + 1, \qquad \rho = 0, 1, \dots, N - 1, \qquad s = 0, 1, \dots \tag{24}$$

Within the sth band, equation (23) is the second-order recurrence with the term $\alpha K_{s-1}(\rho)$ that is determined at the previous step, while finding the correlation function for the $(s - 1)$th band.

3.2.2. General expression for the correlation function. In the zeroth band ($s = 0, 1 \leqslant r \leqslant N$), as was shown above, the correlation function is constant:

$$K_0(\rho) = K_0. \tag{25}$$

For the first band ($s = 1, N + 1 \leqslant r \leqslant 2N$), considering that $K(r - N - 1) = K_0(\rho)$, we have

$$K_1(\rho) = (1 - (1 - \alpha N)(1 + \alpha)^{\rho})K_0. \tag{26}$$

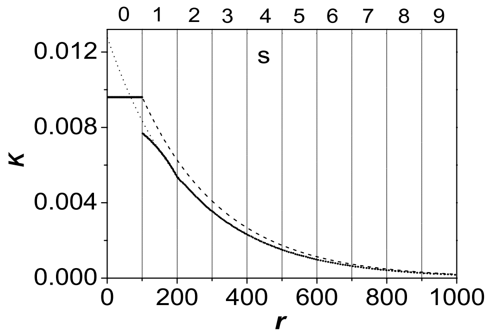

Figure 2. Correlation function $K(r)$ (solid line) obtained by two different methods: the numerical simulation of equation (14) and exact solution equation (41). The dotted line is for the contribution to $K(r)$ of the first root of equation (32). The dashed line refers to the correlation function obtained in [19]. The vertical lines indicate the limits of the bands numbered by s.

The correlation function decreases quasi-continuously within the first band[4]. However, as was mentioned above, there exists a discontinuity in the $K(r)$ dependence at $r = N$. This discontinuity disappears in the limiting case of the strong persistence, $\alpha \to 1/N$.

Substituting equation (26) into equation (23), we find the solution $K_2(\rho)$ for the second band ($s = 2, 2N + 1 \leqslant r \leqslant 3N$):

$$K_2(\rho) = (1 - (1 - \alpha N)((1 + \alpha)^{\rho+N} - \rho\alpha(1 + \alpha)^{\rho-1})K_0. \tag{27}$$

The correlation function $K(r)$ is continuous at the interface between the first and second bands, $K_1(N) = K_2(0)$. However, its first derivative of $K(r)$ is discontinuous here (see figure 2). Using the induction method, one can easily derive the formula for $K_s(\rho)$ in the sth band ($sN + 1 \leqslant r \leqslant (s + 1)N$):

$$\begin{aligned} K_s(\rho) &= \left(1 - (1 - \alpha N)\sum_{i=1}^{s}(-\alpha)^{i-1}(1 + \alpha)^{(s-i)N+\rho-i+1}C_{(s-i)N+\rho}^{i-1}\right)K_0, \\ C_n^k &= \frac{\Gamma(n + 1)}{\Gamma(k + 1)\Gamma(n - k + 1)}. \end{aligned} \tag{28}$$

It follows from equation (28) that the first $(s - 2)$ derivatives of the correlation function $K(r)$ are continuous at the border between the $(s - 1)$th and sth bands, but the derivative of the $(s - 1)$th order changes discontinuously. With $\alpha N \ll 1$, equation (28) takes a simpler form

$$K_s(\rho) = K_0\alpha^s C_{s+N-1-\rho}^s. \tag{29}$$

It is seen that the correlation function decreases proportionally to α^s with an increase of the band number s.

It is not easy to analyse the asymptotical behaviour of the function $K(r)$ at large s because the number of summands in equation (28) increases proportional to s. This is a good reason to propose another approach to the asymptotical study of the correlation function $K(r)$ at $s \gg 1$.

[4] Denote $A = \alpha N$ and suppose that value A is constant while $N \to \infty$. Let $g(x) = \lim_{N\to\infty} K(xN)$. If the function $g(x)$ is a continuous function, we refer to $K(r)$ as a *quasi-continuous* function. If function $g(x)$ has a point of discontinuity at x_0, we refer to $r = x_0N$ as a *point of discontinuity* of function $K(r)$. In our case, $g(x) = K_0$, at $0 < x < 1$, and $g(x) = K_0(1 - (1 - A)\exp((x - 1)A))$, at $1 < x < 2$.

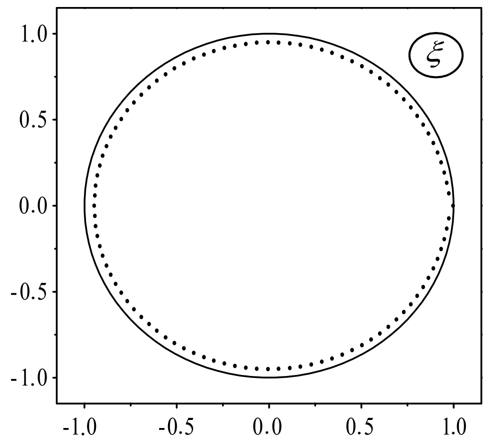

Figure 3. The dots are the roots of the characteristic equation (32) for $N = 100$ and $\alpha = 0.008$. The solid line is the circle $|\xi| = 1$.

3.3. Asymptotical study of the correlation function

3.3.1. Derivation of the characteristic equation. The general solution of linear recursion equation (21) can be represented as the linear combination of N different exponential functions,

$$K(r) = \sum_{i=1}^{N} a_i \xi_i^r. \tag{30}$$

To find the values of ξ_i, we substitute the fundamental solution,

$$K(r) = \xi^r, \tag{31}$$

into equation (21) and obtain the characteristic polynomial equation of the order of N. Constant multipliers a_i should be determined by initial conditions.

It is more convenient to use equation (23) instead of equation (21), that leads to the characteristic equation of the order of $N + 1$:

$$\xi^{N+1} - (1+\alpha)\xi^N + \alpha = 0. \tag{32}$$

The extra root of this equation, $\xi = 1$, appears as a consequence of passing on to the equation of order of $N+1$ from that of the order of N. The corresponding coefficient, a_i, in equation (30) is equal to zero because the correlation function should decrease at $r \to \infty$.

Our study shows that equation (32) has one real positive root less than unity in the case of odd N. In the case of even N, there are two real roots, one positive and one negative. The rest of the roots are complex. All absolute values of roots are less than unity, which is in agreement with the finiteness of the memory function $F(r)$. In the case of large N, the absolute magnitudes of all roots are close to unity for nearly all values of α satisfying the inequality,

$$\frac{1}{N} \ln \frac{1}{\alpha} \ll 1. \tag{33}$$

The distribution of the roots in the complex plane ξ is shown in figure 3.

In the simplest case, $N = 2$, equation (32) has two real roots:

$$\xi_{1,2} = \frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \alpha}. \tag{34}$$

Taking into account the initial conditions, we find the solution of equation (21) in the form

$$K(r) = \frac{\alpha}{4(1-\alpha)\sqrt{\alpha^2+4\alpha}}\big(\xi_1^{r-1}(1-\xi_2) - \xi_2^{r-1}(1-\xi_1)\big). \tag{35}$$

This expression can be simplified at small and large values of parameter α. For $\alpha \ll 1$, one obtains

$$K(r) = \tfrac{1}{4}\alpha^{[r/2]+1} \tag{36}$$

with square brackets standing for the integer part. The correlation function in the sequential odd and even points is equal to each other. In accordance with equation (29), $K(r)$ decreases at $r \to \infty$ in proportion to α^s. In the opposite limiting case of the strong persistency, $\alpha \to 1/2$, we have two different roots:

$$\xi_1 = 1 - \tfrac{4}{3}\phi, \qquad \xi_2 = -\tfrac{1}{2} + \tfrac{1}{3}\phi, \tag{37}$$

with $\phi = 1/2 - \alpha$. The coefficient corresponding to the second root is much less than that corresponding to the first one. Besides, the second term in equation (35) decreases more rapidly. Therefore, the approximate solution in this case is

$$K(r) = \tfrac{1}{4}\exp(-4\phi(r-1)/3). \tag{38}$$

3.3.2. Correlation function at small α. Let us return to the case of an arbitrary value of N. If α is very small, i.e. at

$$\frac{1}{N}\ln\frac{1}{\alpha} \gg 1, \tag{39}$$

Equation (32) has N roots with small absolute magnitudes:

$$\xi_k = \alpha^{1/N}\left(\cos\left(2\pi\frac{k}{N}\right) + \mathrm{i}\sin\left(2\pi\frac{k}{N}\right)\right), \qquad k = 0,\ldots,N-1. \tag{40}$$

The correlation function, being a linear combination of the power functions with these roots as their exponents, decreases proportionally to α^s, which agrees with equation (29).

The coefficients a_i in the linear combination in equation (30) can be found in the general case, without any restrictions imposed on the value of α. The solution of equation (21) written for $1 \leqslant r \leqslant N-1$ along with $K(0) = 1/4$ can be expressed by means of the Vandermond determinants:

$$K(r) = K_0(\alpha N - 1)\sum_{k=1}^{N}\frac{\xi_k^{r-1}}{\prod_{j=1, j\neq k}^{N+1}(\xi_k - \xi_j)}, \tag{41}$$

with $\xi_{N+1} = 1$.

3.3.3. Correlation function at not too small α. In the case (33) of not too small α, the absolute magnitudes of all roots are close to unity. It is convenient to rewrite equation (32), introducing two new real variables γ and φ instead of complex x according to

$$x = \left(1 - \frac{1}{\gamma N}\right)\mathrm{e}^{\mathrm{i}\varphi}. \tag{42}$$

Equation (32) takes the form

$$\alpha\big(\mathrm{e}^{\frac{1}{\gamma} - \mathrm{i}N\varphi} - 1\big) = \left(1 - \frac{1}{\gamma N}\right)\mathrm{e}^{\mathrm{i}\varphi} - 1. \tag{43}$$

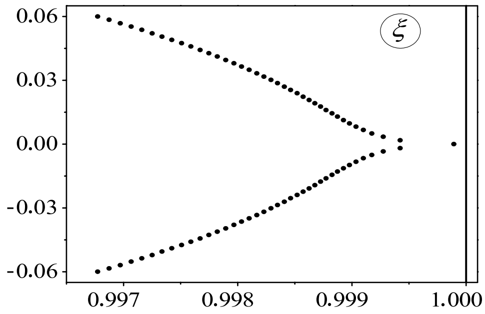

Figure 4. The roots of the characteristic polynomial equation close to the point $\xi = 1$ for $N = 4000$, $\alpha = 2 \times 10^{-4}$. The solid line is $\mathrm{Re}\,\xi = 1$.

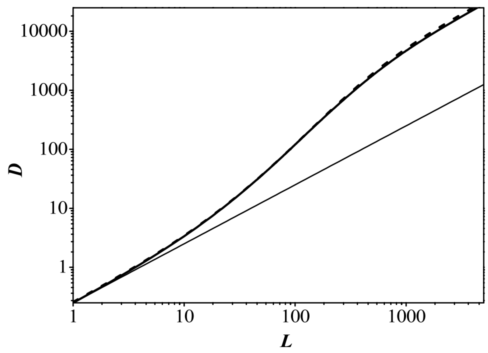

Figure 5. Variance $D(L)$ for the Markov chain with $N = 100$, $\alpha = 0.008$ calculated by means of exact equations (41), (5) (solid line) and solely using one root of equation (32) (dashed line). The thin solid line describes the non-correlated Brownian diffusion, $D(L) = L/4$.

For the real root, equation (32) yields

$$\alpha N \gamma (\mathrm{e}^{1/\gamma} - 1) = 1. \tag{44}$$

This expression along with equation (31) determines the asymptotical behaviour of the correlation function. It was first obtained in [19]. The qualitative approach allowed the authors of [19] to obtain the correct expression for the exponential rate coefficient γ. But an incorrect assumption about the quasi-continuity of the correlation function on the border between the zeroth and first bands was made. It produced a wrong multiplier before $\exp(-\gamma r)$. Besides, this approach yielded the incorrect behaviour of the correlation function on the first bands.

Equation (43) gives all remaining complex roots with the values of φ, which are quite uniformly distributed over the circle $[0, 2\pi]$ and

$$\frac{1}{\gamma} \sim \ln \frac{1}{\alpha}. \tag{45}$$

The roots of equation (32) located in the vicinity of point $\xi = 1$ are shown in figure 4. The single real root is much closer to the line $\mathrm{Re}\,\xi = 1$ than the other ones. Besides, the coefficients a_i in equation (41) (see also equation (30)) for all terms containing the complex exponents are much less than those for the term with the real exponent. Therefore, the behaviour of the correlation function $K(r)$ is generally determined by the term with the real exponent.

The exact correlation function $K(r)$ resulting from the numerical simulation of equation (41) and its approximation determined by the contribution of the real root alone are shown in figure 2. These curves are compared with the curves presented in [19] by a qualitative method.

The correlation function thus obtained can be used to calculate one of the most important characteristics of the random binary sequences, the variance of the number of unities in the L-word. The results of the numerical simulations are shown in figure 5. One can see a good agreement of curves plotted using both of these methods.

4. Conclusion

In summary, we have demonstrated the efficiency of describing the symbolic sequences with long-range correlations in terms of the many-step Markov chains with the *additive* memory function. Actually, the memory function appears to be a suitable informative 'visiting card' of any symbolic stochastic process. Various methods for finding the memory function via the correlation function of the system are proposed. Our preliminary consideration suggests that it is possible to generalize our concept of the Markov chains to a larger class of random processes where a random variable can take on an arbitrary, finite or infinite number of values.

The proposed approach can be used to analyse different correlated systems in the diverse fields of science. For example, the application of the Markov sequences to the theory of spin chains with long-range interaction enables one to estimate some thermodynamic characteristics of these non-extensive systems.

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